Collective motion occurs inevitably in a class of populations of globally coupled chaotic elements

Naoko Nakagawa
Laboratory for Information Synthesis, RIKEN Brain Science Institute, 2-1 Hirosawa, Wako, Saitama 351-01, Japan

Teruhisa S. Komatsu
Department of Pure and Applied Sciences, College of Arts and Sciences, University of Tokyo, Tokyo 153, Japan
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We discovered numerically a scaling law obeyed by the amplitude of collective motion in large populations of chaotic elements. Our analysis strongly suggests that such populations generically exhibit collective motion in the presence of interaction, however weak it may be. A phase diagram for the collective motion, which is characterized by peculiar structures similar to Arnold tongues, is obtained. [S1063-651X(98)06502-7]

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I. INTRODUCTION

Systems consisting of large populations of interacting dynamical elements are widely distributed in nature from communities of ants to biological cell assemblies and neural networks. Many such populations are similar in the respect that they exhibit various kinds of collective behavior. One possible approach to the mathematical study of such collective behavior is to concentrate on certain idealized models such as interacting limit cycles and chaotic maps. An abundance of literature has been devoted to the study of these models [1–9]. It is suggested in some foregoing studies [6–9] that a population of this type as a whole exhibits low-dimensional behavior. This seems to be true even when the individual elements appears to be mutually uncorrelated, a situation in which it would seem that the population could only be regarded as a dynamical system of extremely high dimension. It still remains unclear whether such collective behavior could be understood in terms of low-dimensional dynamical systems.

Without directly attacking the above-stated problem, we take in the present paper a slightly different approach in which we focus on a nonspecific property of collective motion, a long-time average of the amplitude of collective motion. We find that this approach, with which we intentionally avoid addressing specific features of collective motion, turns out to be extremely useful. As model systems we have chosen globally coupled tent maps because these systems are particularly well suited for detailed numerical analysis. Using these maps, we have discovered a scaling law characterizing the amplitude of collective motion that holds for a particular series of parameter values. Further, we argue on the basis of numerical evidence that this type of collective motion should occur generically if an interaction exists, no matter how weak this interaction may be. Finally, a phase diagram for the collective motion is obtained and the existence in it of peculiar structures similar to Arnold tongues with various scales is confirmed.

II. MODEL: GLOBALLY COUPLED TENT MAPS

Globally coupled maps (GCMs) are given by an assembly of N elements whose behavior is determined by N identical maps with all-to-all coupling. The individual elements are then under the influence of a common internal field, which may be referred to as a mean field [5]. We assume that a single isolated element evolves according to $X_{n+1}=f(X_n)$, where n designates discrete time steps. Under the interaction through the mean field h_n, the ith element is then assumed to evolve as

$$X_{n+1}^{(i)}=f(X_n^{(i)})+Kh_n, \tag{1}$$

where K is the coupling strength. In this paper we consider the situation in which f is a tent map

$$f(X)=-a|X|+\frac{a-1}{2} \tag{2}$$

and the mean field h_n is defined as

$$h_n\equiv\frac{1}{N}\sum_{j=1}^{N} f(X_n^{(j)}). \tag{3}$$

Thus our system is characterized by the two parameters a and K in addition to the total number of elements N. Each tent map has a band splitting point $a=\sqrt{2}$ and we thus stipulate that a satisfies $\sqrt{2}<a<2$. It is thereby ensured that the population will never split into subpopulations. If the system size N is a finite, this finiteness becomes the source of fluctuations of the mean field. Such an effect may obscure pure collective motion. Thus we work with the limit of large N. In fact, we confirmed that finite-size effects can be regarded as noise acting on the pure collective motion. For $N\to\infty$, the population dynamics of GCMs can be described by the Frobenius-Perron equation [11] for the distribution $\rho(X)$:

$$\rho_{n+1}(X)=\int \delta(X-f(X')-Kh_n)\rho_n(X')dX', \tag{4}$$

$$h_n=\int f(X')\rho_n(X')dX'. \tag{5}$$

We worked out a numerical scheme for the exact integration of Eqs. (4) and (5) whose precision is limited only by round-

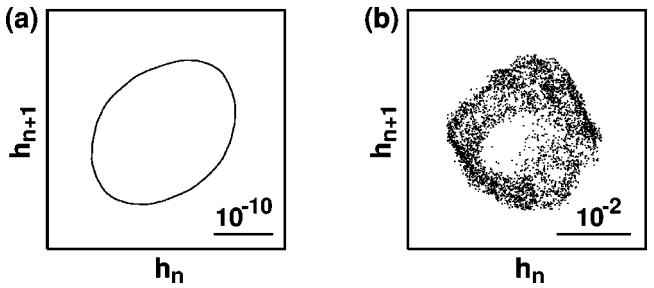

FIG. 1. Two types of collective motion at $a=g_5$ are displayed by the return map of h_n. (a) For $K=0.1$, we find a torus representing quasiperiodic motion. (b) For $K=1.0$, the return map indicates more complicated toruslike motion possibly accompanied by fine structure.

off errors. Our scheme is almost the same as that proposed by Morita [10], although these numerical schemes were developed independently. We have confirmed that collective motion appearing in the system described by Eqs. (4) and (5) is almost independent of the initial distribution. For this reason we present numerical results in this paper that were obtained using only a single initial distribution, uniform over the interval $[f\circ f(0), f(0)]$.

III. COLLECTIVE MOTION IN GLOBALLY COUPLED TENT MAPS

In some previous works [7,11], it has been argued that the model constituted by Eqs. (1) and (2) does not exhibit collective motion. However, our careful analysis yields a contradictory conclusion. Collective motion can be observed through the dynamics of the order parameter h_n given by Eq. (5). Figures 1(a) and 1(b) are return maps of h_n. There we see that the fluctuation of h_n undergoes quasiperiodic motion for small K, but for larger K it displays more complicated motion possibly with fine structure. Roughly speaking, larger K values results in more complicated collective behavior.

The observed collective motion is similar to that found in other models [6–8], except that the scale of the collective motion for the present model is much smaller. Because of this smallness, the previous works [7,11] have failed to detect the collective motion for the present model. Based on their conclusion that collective motion does not exist here, they conjectured that the presence of window structures for the elements (which tent maps never possess) are necessary for the occurrence of collective motion, but our results shows this is not the case.

IV. PARAMETER DEPENDENCE OF THE COLLECTIVE MOTION: HILLY STRUCTURE

We concentrate on the amplitude F of the collective motion without going into any detailed structure of the dynamics. We have

$$F\equiv\sqrt{\langle(h_n-\langle h_n\rangle)^2\rangle}, \tag{6}$$

where $\langle\ \rangle$ represents a long-time average. In Fig. 2 the dependence of F on a is shown for two values of K. The horizontal axis represents $m\equiv-\log_a(2-a)$, which is a monotonically increasing function of a on [0,2], tending to

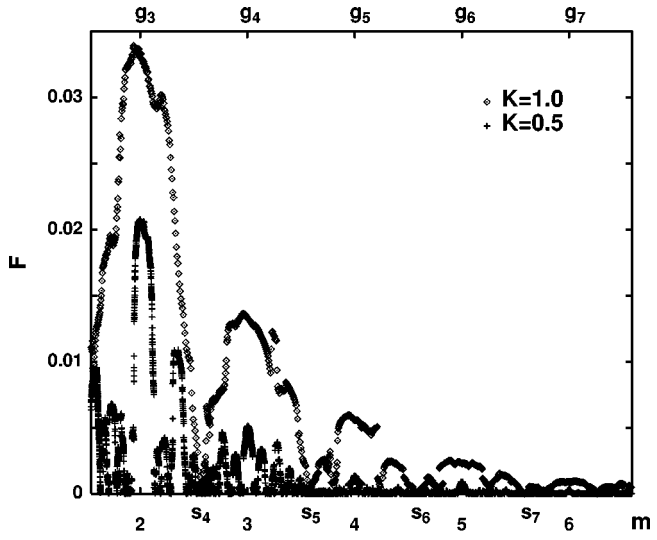

FIG. 2. Amplitude of collective motion F vs m for two values of K. Here $m=-\log_a(2-a)$. There are several hills and valleys, whose numbers increase as K decreases. Some golden and silver values, denoted by g_p and s_p, are also displayed for reference.

infinity as $a\to2$. The figure shows the range of many hills of F, which we call a ''hilly structure.'' As the value of K is decreased, the number of the hills becomes larger, while the average size of each hill becomes smaller. For larger a (i.e., larger m), the hilly structure is difficult to discern in this figure, but its persistence can be confirmed by magnifying the scale. The hilly structure seems to exist for arbitrarily small values of K. Furthermore, based on this figure, we expect that for a given value of K, collective motion exists for almost all a. These points will be discussed in further detail below.

V. GOLDEN AND SILVER VALUES OF PARAMETER a

We find that a particular series of parameter values of a [12], which we call ''golden values,'' play a crucial role in the formation of such a hilly structure. We find in Sec. VII that these parameter values are situated in the middle of each hill for sufficiently small values of K. Some examples are indicated in Fig. 2 with the notation g_3–g_7. When a is identical to one of the golden values, one isolated tent map possesses the property that a trajectory beginning at the peak C of the tent map returns to C after p steps. We denote a golden value defined in this way as g_p. Incidentally, we use the term ''golden value'' because the value g_3 equals the golden mean.

We refer to another special series of parameter values of a as ''silver values.'' These are associated with the valleys between neighboring hills, as argued in Sec. VIII. For instance, the silver values s_4–s_7 are indicated in Fig. 2. When a is identical to s_p, a trajectory beginning at C falls into a fixed point of the tent map after p steps. Golden and silver values exist densely if we allow p to take all natural number values, while they occupy only a vanishing measure on the line of a. We note that g_p and s_p are generally not unique because of the multiplicity of p-periodic orbits, whose number increases almost exponentially with p. Despite this nonu-

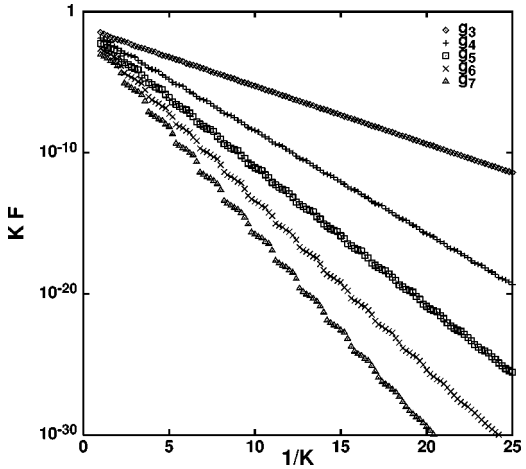

FIG. 3. Scaling of F with K. Here KF vs $1/K$ is shown for five golden values g_p, $p=3-7$. The values of g_p are the same as those in Fig. 2. The data clearly exhibit the linear dependence of $\log_{10}(KF)$ on $1/K$.

niqueness, $g_p(s_p)$ is used below to represent a single golden (silver) value.

VI. SCALING LAW OF COLLECTIVE MOTION AT GOLDEN VALUES

We now consider the collective motion as a function of the coupling strength K, with a confined to golden values. We found numerically that the amplitude F of the collective motion obeys the law

$$KF \sim e^{-\alpha/K}, \tag{7}$$

where α is a positive constant depending on the golden values in question. For any golden value, the above scaling form seems to hold as long as K is sufficiently small. In Fig. 3 the relation between KF and K^{-1} is displayed for those golden values that are situated in the middle of some representative hills in Fig. 2. The formula in Eq. (7) reveals some important properties of the collective motion. First, the collective motion persists even as $K \to 0$. Second, the observed quasiperiodic motion does not appear through the conventional route of Hopf bifurcation. This follows from the fact that if the quasiperiodicity were due to a Hopf bifurcation, then we would find that $F \sim \sqrt{|K-K_c|}$ above a bifurcation point K_c. This is clearly in contradiction with Eq. (7). In our study we could not even find an indication of a bifurcation. Third, the simple relation in Eq. (7) survives even when the collective motion becomes more complicated. Note that different modes of collective motion, as illustrated in Figs. 1(a) and 1(b), lie on a common line of Fig. 3.

VII. FORMATION OF A HILLY STRUCTURE

The appearance of a hilly structure, seen in Fig. 2, can be understood from two arguments concerning hills and valleys. The following discussion regarding hills based on Eq. (7) partially explains how collective motion comes to characterize almost all values of a, not just the golden values.

At a given value of K each hill of F contains a number of golden values. One of these golden values is found to be

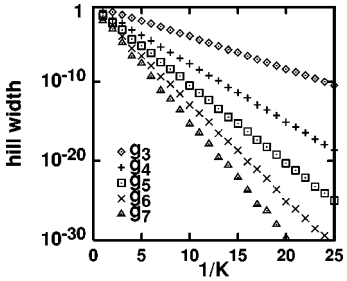

FIG. 4. Widths of the hills around a few key golden values. Here the hill width vs $1/K$ is shown for five golden values g_p ($p=3-7$) whose values are the same as those in Fig. 3. For each golden value, we find a good correspondence between the line here and that in Fig. 3.

representative of this hill. We call this a key golden value. In order to define key golden values, we studied the growth of the width of hills with the decrease of K. Figure 4 shows how the width W of the hill around a given golden value changes with K^{-1}. The width W around a given g_p is defined by $W=|a_L-a_R|$, where a_L and a_R are the two values of a on either side of g_p such that the amplitude F at $a=a_L$ and $a=a_R$ is half of $F(g_p)$. The golden values chosen here are the same ones as in Figs. 2 and 3. In Fig. 4 we see that W behaves as a function of K similar to that in Eq. (7). Actually, the width W for a given g_p is proportional to KF calculated for the same g_p, as long as K is sufficiently small. This implies that the hill narrows down to a point, corresponding to $a=g_p$. Based on this relation, we define key golden values of a given hill that exists at a given K_0 as follows: With the decrease of K from K_0, the width of a hill narrows down to a single point according to the growth law for W. The value of a for this single point is that of the key golden value of this hill. The definition of key golden values will be refined as golden values displaying the scaling law (7) for values of K smaller than K_0.

The formation of the hills may roughly be understood from the following argument. At a golden value g_p, the invariant measure for a single map, i.e., $\rho(X)$ at the equilibrium solution of the Frobenius-Perron equation with $K=0$, has the rather simple form [12] of a $p-2$ step function, as displayed in Fig. 5(a). As a deviates slightly from g_p, the

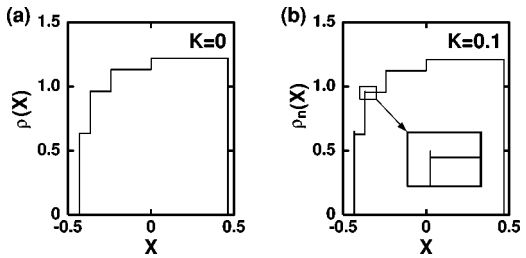

FIG. 5. (a) Invariant measure $\rho(X)$ and (b) snapshot of the distribution $\rho_n(X)$ at a certain time step n, where a is set at the golden value g_5. (a) The invariant measure is given by a three-step function. (b) The instantaneous shape of the distribution for $K=0.1$ is similar to the invariant measure, except that it is accompanied by peaks with finite widths that appear near each corner of the steps. The snapshots of $\rho_n(X)$ for all values of a in the neighborhood of g_5 have a similar property.

change of the invariant measure remains small, although there may appear some fine structure. When a weak interaction is switched on, the solution $\rho_n(X)$ of Eqs. (4) and (5) with $a=g_p$ continues to assume a shape quite similar to the invariant measure $\rho(X)$ mentioned above, but it is accompanied by intermittent peaks [see Fig. 5(b)] whose widths are roughly given by KF. The situation remains almost the same when a differs slightly from g_p. If the effect of these peaks on the amplitude of the collective motion is more important than the change in the invariant measure caused by a small deviation of a from g_p, the statistical average F will be insensitive to the value of a around g_p. If the deviation of a from g_p becomes too large, however, the corresponding F will be dominated by the influence of other golden values corresponding to other hills.

Hills formed in the manner described above are consistent with hills obeying the growth relation of the width W (see Fig. 4). Further, the above reasoning applies to every golden value. Thus every golden value can be considered as a key golden value around which its own hill is formed. This conclusion may appear to be inconsistent with the fact that in a given hill there are an infinite number of golden values. Actually, for a given K, most golden values in a given hill are *not key golden values. We have found that around such a K value, the amplitude F for these golden values does not obey the scaling law of F. This will be discussed in Sec. IX. It is now clear that collective motion is not confined only to values corresponding to golden values. It occurs over the intervals corresponding to hills existing around these golden values.*

VIII. COLLECTIVE MOTION IS DOMINANT: PROPERTY AROUND VALLEYS

The hilly structure seen in Fig. 2 can be understood by analyzing the structure of F in the valleys between two neighboring hills. From this analysis we are able to conclude that collective motion occurs generically in the a-K plane. We specifically investigate the parameter dependence of F in valleys. As is seen from Fig. 2, we find that some silver values lie precisely at the minimum points of F in the valleys. Figure 6 shows that when the value of a is near a silver value s_p, we obtain the following relation for sufficiently small values of K:

$$F \propto |a-s_p|^{\beta} \quad (\beta \simeq 1). \tag{8}$$

It is clear from this relation that collective motion disappears only at the point $a=s_p$ [13]. This relation seems to hold for any silver value when the value of K is sufficiently small. This fact strongly suggests that the total measure of values of a at which collective motion does not exist is vanishingly small because the total measure corresponding to silver values is vanishing, although they exist densely on the line of a. The remaining values of a are characterized by hills that extend from key golden values with widths that increase with K. Hence the occurrence of the collective motion is dominant over the parameter space of a. Because of the scaling law existing at golden values, this situation holds even for vanishingly small values of K.

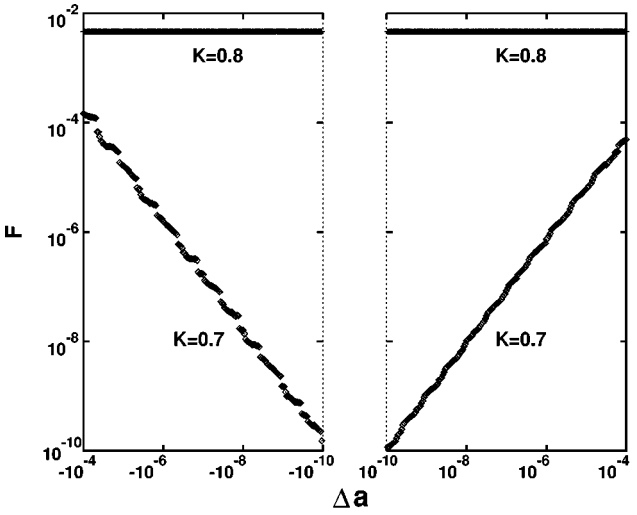

FIG. 6. Amplitude of collective motion F near a silver value s_7 displayed on a logarithmic scale, where $\Delta a \equiv a - s_7$. The picture on the left is for values of a smaller than s_7, while that on the right is for a larger than s_7. For smaller K, $F \propto |\Delta a|$, so that F decays to zero as $a \to s_7$. On the other hand, F remains finite for larger K over the range of a in this figure.

IX. MERGING OF HILLS

Figure 6 also indicates that the valleys disappear for sufficiently large values of K. For each silver value, we find a critical value K_c for the disappearance of the corresponding valley. When K exceeds such a K_c, a valley that existed at a certain silver value up to $K=K_c$ suddenly disappears. This occurs because two hills lying on either side of this silver value merged into one at $K=K_c$. As a result, the amplitude F at this value of a exhibits a sudden increase. As described above, each hill possesses a key golden value. After the merging of two hills, one of the two key golden values characterizing the previously existing hills, say g_{p_1}, ceases to be a key golden value, while the other persists. As a result the K dependence of F at $a=g_{p_1}$ changes and typically comes to behave as illustrated in Fig. 7. We see in this figure that the

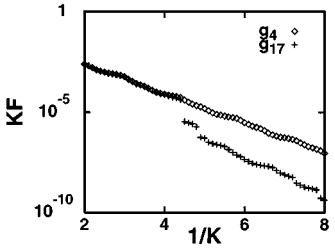

FIG. 7. KF vs K^{-1} for two golden values whose values of a differ only slightly. The plotting method is the same as that for Fig. 3. The value of g_4 is the same as that given in Fig. 2. Each golden value is regarded as a key golden value, while K is sufficiently small. However, only the collective motion at one golden value g_4 is characterized by the same scaling law in both the small-K and large-K regions. The K^{-1} dependence of KF at the other golden value g_{17} changes and this dependence comes to resemble that at g_4. This implies that the hill associated with g_4 absorbs the hill associated with g_{17}. Thus the golden value g_4 continues to be a key golden value, while g_{17} ceases to be such.

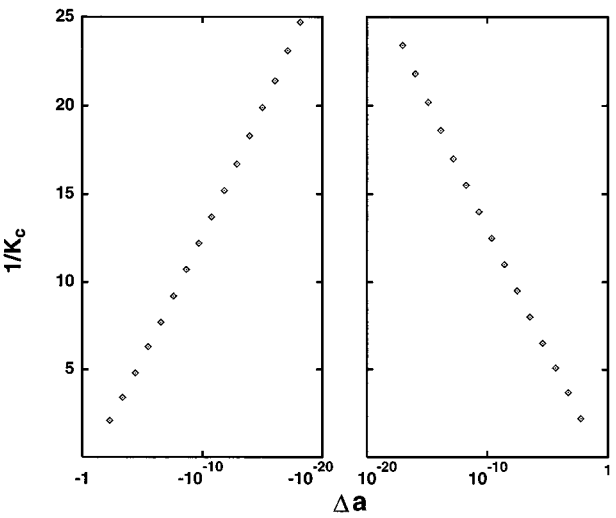

FIG. 8. The point K_c at which the minimum point in the valley closest to the key golden value g_4 vanishes is investigated. Here $\Delta a = a - g_4$. The value of g_4 is the same as that given in Fig. 2. Note that the horizontal scale is logarithmic. The picture on the left is for the values of a smaller than g_4, while that on the right is for larger values. When a minimum situated at the silver value disappears at some $K = K_c$, the width of the hill associated with g_4 becomes expanded. The values of K_c for those valleys that disappear successively show a logarithmic dependence on Δa. Although this figure ignores many other valleys that are actually present, we believe that the manner in which valleys disappear successively is consistent with the variation of the hill width with K shown in Fig. 4.

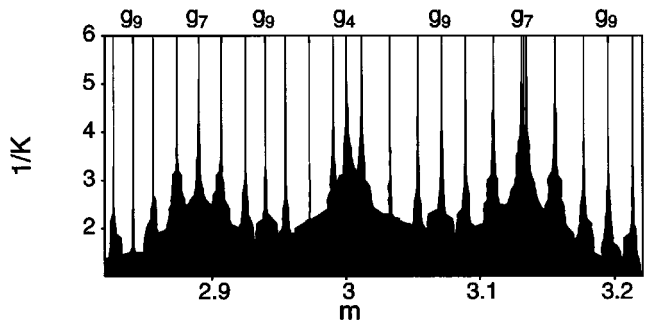

FIG. 9. Blowup of the phase diagram for the collective motion in m-K^{-1} space, where $m = -\log_a(2-a)$ (restricted in $2.82 < m < 3.22$). The value of the g_4 in this figure is the same as that in Fig. 2. Each domain filled with black has been determined from the calculation of the hill width associated with the corresponding golden value. The golden values represented here are g_p with $p \leqslant 11$. The neighboring domains overlap for larger values of K. A precise phase diagram split into an infinite number of domains could be imagined as an extension of this figure.

K dependence of $F(g_4)$ is maintained, while that of $F(g_{17})$ changes above K_c, where it comes to exhibit a K dependence similar to $F(g_4)$.

The merging of two hills leads to a sudden expansion of the width of one of the hills, accompanied by the disappearance of the other. If a hill associated with some key golden value g_p successively absorbs the neighboring hills with the increase of K, then the width W of this hill also grows successively. On the other hand, we have already seen in Fig. 4 a growth law for W. These observations suggest that the values of K_c characterizing the disappearance of valleys around silver values should be correlated with our growth law for W. In the case of Fig. 8, the golden value g_4 persists as a key golden value over a wide range of K. We calculated the values of K_c for a series of silver values. Each of these silver values was chosen in such a way that in a certain range of K it lies at the minimum point between the hill associated with g_4 and a neighboring hill. In fact, there should exist many more silver values of this kind for a given g_4 than those displayed in this figure. We see from this figure that the value of K_c for a given silver value depends on the distance of this silver value from g_4 and that this dependence is consistent with the growth law for W seen in Fig. 4. In many cases, merging occurs between hills of vastly different sizes. In this case, the collective motion characterizing the smaller hill seems to be replaced suddenly by that for the larger one. In such cases, the hill width may appear to grow almost continuously.

From our study presented so far, the hilly structure of the amplitude F of the collective motion can be understood. This structure consists of a series of hills and valleys. Only at a minimum point of F in each valley, collective motion disappears. Thus the hilly structure implies that under a given coupling strength K the collective motion occurs inevitably for almost all values of a. Furthermore, the following properties for the collective motion are now clear: There is a scaling law for the amplitude F, a growth law for the hill width W, and a successive merging of hills resulting in a decrease in the number of the hills and hence the number of key golden values. Clearly these features are interrelated.

X. PHASE DIAGRAM

We now consider a phase diagram of collective motion in the a-K^{-1} plane from the viewpoint of the amplitude F. For each golden value g_p, we define a phase as the region of a hill (defined by values of F greater than some threshold value F_{th}) associated with a certain g_p. Here the definition is meaningful only if g_p represents a key golden value. Since golden values exist densely, such a diagram will be split into an infinite number of domains as $K \to 0$, each reducing to a single point corresponding to a golden value. The extension of each domain becomes larger with the increase of K, as implied by Eq. (7), and this necessarily results in the successive merging of domains, leaving only a few for sufficiently large K. Thus our phase diagram resembles that for the phase locking between oscillators. The latter is also characterized by the merging and splitting of domains, in this case synchronized domains, referred to as Arnold tongues. We note, however, there is one property of our phase diagram that distinguishes it from that in the case of phase locking. In the present case, in the $F_{th} \to 0$ limit, the domains of collective motion comprise all but a measure 0 set of values of a, even for vanishingly weak coupling.

In Fig. 9 we show a blowup of the phase diagram that contains only those domains of collective motion for g_p with $p \leqslant 11$. The numerical procedure for obtaining this picture essentially follows that for Fig. 4. We determined the domain for each g_p with a given K first, displaying it as a painted region of a. We then did the same for various K, thereby

producing Fig. 9. Note that the domains separated for a given K correspond to distinct hills with distinct g_p. As expected, a pair of neighboring domains merge at a certain value of K at which a valley disappears. Thus the number of separated domains decreases with the increase of K. After several merging events, we are left with only one painted domain associated with g_4.

XI. DISCUSSION: ANY UNIVERSALITY CLASS?

The foregoing arguments suggest that collective motion occurs generically for arbitrarily weak coupling. Such a result seems to be closely related with some global properties of the individual maps such as the topological arrangement of golden values. It is our conjecture that the same conclusion holds for such globally coupled unimodal maps satisfying $|f'(X)|>1$ for any X. Although not reported in the present paper, we also confirmed that a similar scaling law holds for a variety of mean fields h_n. From these studies it is seen that the analytical approach by Ershov and Potapov [14] gives estimated values of F that are much smaller than those found in the present numerical results.

For the case not satisfying the above condition for $|f'(X)|$, for instance, globally coupled logistic maps, the present argument is expected to be qualitatively applicable under some restrictions. This is mainly because here golden values are located in windows and the stability of a periodic orbit there causes the so-called clustering phenomena [5], which are beyond the present arguments. On the other hand, collective motion resembling the present motion is also observed for globally coupled logistic maps, where the amplitude of this collective motion is proportional to the coupling strength [11]. An analytical study [15] also supports this idea. These suggests the existence of a certain scaling law of F that implies the qualitative applicability of our argument.

Collective motion with a quasiperiodic property is also observed in spatially extended systems [6]. Since the GCM is an idealized model of a long-range coupled system, it does not take into account anything resembling spatial extension. We wish to be able to find a way to include the effects of spatial degrees of freedom in the description of collective motion. It is hoped that the discovery presented here will be relevant to such spatially extended systems also.

We have concentrated on the amplitude F of the collective motion without going into its phase-space structure. The scaling relation discovered for this nonspecific quantity holds over a wide range of parameter values for which the type of collective motion changes in various ways. Based on this point, we suspect that the collective motion discussed in the present paper might possess quite unusual properties not shared by conventional low-dimensional dynamical systems. Detailed numerical study of the nature of various types of collective motion, which is now in progress, will clarify this point.

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